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Deformed Kac–Moody and Virasoro algebras

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Abstract

Whenever the group $\mathbb{R}^n$ acts on an algebra $\mathcal{A}$, there is a method to twist $\mathcal{A}$ to a new algebra $\mathcal{A}_\theta$ which depends on an antisymmetric matrix θ ($\theta^{\mu\nu} = -\theta^{\nu\mu} =$ constant). The Groenewold–Moyal plane $\mathcal{A}_\theta(\mathbb{R}^{d+1})$ is an example of such a twisted algebra. We give a general construction to realize this twist in terms of $\mathcal{A}$ itself and certain 'charge' operators Q_μ. For $\mathcal{A}_\theta(\mathbb{R}^{d+1})$, Q_μ are translation generators. This construction is then applied to twist the oscillators realizing the Kac–Moody (KM) algebra as well as the KM currents. They give different deformations of the KM algebra. From one of the deformations of the KM algebra, we construct, via the Sugawara construction, the Virasoro algebra. These deformations have an implication for statistics as well.

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1. Introduction

1.1. Preliminaries

Let (M, g) be a Riemannian manifold with a metric g. Suppose that $\mathbb{R}^N (N \geqslant 2)$ acts as a group of isometries on M. Then $\mathbb{R}^N$ acts on the Hilbert space $L^2(M, \mathrm{d}\mu_g)$ of square-integrable functions on M. The volume form $\mathrm{d}\mu_g$ for the scalar product on $L^2(M, \mathrm{d}\mu_g)$ is induced from g. Hence this action of $\mathbb{R}^N$ is unitary.

We are interested just in the unitarity of the $\mathbb{R}^N$ action. Hence we can weaken the isometry condition. That is because the action of $\mathbb{R}^N$ remains unitary if it leaves only $\mathrm{d}\mu_g$ invariant without necessarily leaving the metric g itself invariant.

Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ label the unitary irreducible representations (UIRs) of $\mathbb{R}^N$. Then we can write

$$L^2(M, \mathrm{d}\mu_g) = \bigoplus_{\lambda} \mathcal{H}^{(\lambda)}, \tag{1}$$

where $\mathbb{R}^N$ acts by the UIR λ, or in the case of multiplicity, by direct sums of the UIR λ, on $\mathcal{H}^{(\lambda)}$. $\mathbb{R}^N$ being noncompact, we may have to write the direct sum as the direct integral. But, as this issue is not important here, we will use the summation notation.

1.2. The twist

There is a general way of twisting the algebra of functions on M using the preceding structure. We now explain it.

If $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$ and the group law is addition, we choose λ such that $\lambda : a \to \mathrm{e}^{\mathrm{i}\lambda\cdot a}$. Let f_λ and $f_{\lambda'}$ be two smooth functions in $\mathcal{H}^{(\lambda)}$ and $\mathcal{H}^{(\lambda')}$, then under the pointwise multiplication

$$f_\lambda \otimes f_{\lambda'} \to f_\lambda f_{\lambda'},$$

where

$$(f_\lambda f_{\lambda'})(p) = f_\lambda(p) f_{\lambda'}(p).$$

With p a point of M, we have that

$$f_\lambda f_{\lambda'} \in \mathcal{H}^{(\lambda+\lambda')}. \tag{2}$$

Now suppose that θ ($\theta^{\mu\nu} = -\theta^{\nu\mu} =$ constant) is an antisymmetric constant matrix in the space of UIRs of $\mathbb{R}^N$. We can use it to twist the pointwise product to the *-product $*_\theta$ depending on θ where

$$f_\lambda *_\theta f_{\lambda'} = f_\lambda f_{\lambda'}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\lambda_\mu\theta^{\mu\nu}\lambda'_\nu}. \tag{3}$$

The resultant algebra $\mathcal{A}_\theta(M)$ is associative because of (2).

This algebra has been reviewed and developed by Rieffel [1] and many others. $\mathcal{A}_\theta(\mathbb{R}^d)$, the Moyal plane, is a special case of this algebra. In recent times, Connes and Landi [2] and Connes and Dubois-Violette [3] have constructed the full noncommutative geometry for special cases of this algebra.

The above discussion shows that what is pertinent for the twist is not the commutative nature of the underlying algebra. Rather, it is sufficient to have an associative algebra $\mathcal{A}$ graded by $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N (N \geqslant 2)$:

$$\mathcal{A} = \bigoplus \mathcal{A}^{(\lambda)}, \qquad \mathcal{A}^{(\lambda)}\mathcal{A}^{(\lambda')} \in \mathcal{A}^{(\lambda+\lambda')}. \tag{4}$$

Then we can twist it to $\mathcal{A}_\theta$:

$$\alpha_\lambda *_\theta \alpha_{\lambda'} = \alpha_\lambda \alpha_{\lambda'}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\lambda_\mu\theta^{\mu\nu}\lambda'_\nu}, \qquad \alpha_{\lambda,\lambda'} \in \mathcal{A}^{(\lambda),(\lambda')}.$$

We will illustrate this deformation as well using the bosonic creation–annihilation operators and the Kac–Moody (KM) algebra[4].

It is also possible to twist an associative algebra $\mathcal{A}$ to an associative algebra $\mathcal{A}_\theta$ on which $\mathbb{R}^N$ acts even if this action is not unitary. If V_μ, with $\mu = 1, \ldots, N$, is a basis of vector fields for the Lie algebra of $\mathbb{R}^N$, and $\alpha, \beta \in \mathcal{A}$, then the twisted product is the *-product

$$\alpha *_\theta \beta = \alpha\, \mathrm{e}^{\overleftarrow{V}_\mu \theta^{\mu\nu} \overrightarrow{V}_\nu} \beta. \tag{5}$$

[4] In the mathematical literature on KM and Virasoro algebras, their twists are understood differently. (We thank Professor V Dobrev for pointing this out to us.) Hence for these algebras, we replace the phrase 'twist' by the phrase 'deformation'.

However, if the $\mathbb{R}^N$ action is not unitary, we may encounter problems in physical applications.

We remark that the grading lattice in (4) can even be periodic. Thus suppose that

$$\mathcal{A}^{(\lambda_1,\lambda_2,\ldots,\lambda_l+M_l,\lambda_{l+1},\ldots,\lambda_N)} = \mathcal{A}^{(\lambda_1,\lambda_2,\ldots,\lambda_N)},$$

where M_l is the period in direction l. Then it is enough for the consistency that the twisting matrix θ in (3) satisfies

$$\mathrm{e}^{-\frac{\mathrm{i}}{2}M_l\theta^{l\nu}\lambda'_\nu} = 1 \qquad \text{(no } l \text{ sum)}$$

which implies

$$\theta^{l\nu}\lambda'_\nu = \frac{4\pi}{M_l}N, \qquad N \in \mathbb{Z}$$

for each choice of λ'_ν.

1.3. Summary

In section 2, we will apply this construction to a few examples as illustrations. Examples include group and oscillator algebras. In section 3, we realize the twisted algebra in terms of the elements of the original algebra and certain 'charge' operators Q_μ. Such realizations are known in the theory of quantum groups, for instance in the q-oscillator realization of $U_q(su(2))$.

Next in section 4 we construct two different deformed KM algebras. First, we deform the KM algebra using twisted oscillators. Second, we deform the KM generators directly. We then check that the KM algebra remains 'Hopf' after certain deformations as well, but with twisted coproducts. At last we obtain the Virasoro algebra, via the Sugawara construction, using the second form of the deformed KM algebra. The Virasoro algebra we obtain is the same as the usual one.

The deformation affects the R-matrix and the associated braid group representations. This means that the statistics is affected by the twisting as was emphasized in earlier papers [4–8]. The 'Abelian' twist based on $\mathbb{R}^N$ dealt with here does not change the permutation group governing particle identity in the absence of twist. But it does change the specific realization of this group with serious consequences for physics. We discuss the twisted permutation symmetry in section 5.

We have acknowledged certain previous papers [1–3] for the origin of the ideas treated in this paper. It also has overlapped with the Fairlie–Zachos work on 'atavistic' algebras [9]. Our principal concern is the systematic construction of deformed algebras in terms of undeformed ones, which appears to originate from our own previous work.

The method used for the deformation of algebras presented in this paper has similarities to a method used in the works of Hu [10, 11]. In particular, the deformed KM algebras we obtained in the present work are the same as the 'colour' KM algebra obtained in [11].

2. Examples of Abelian twists

As mentioned in the introduction, (M, g) is a Riemannian manifold on which $\mathbb{R}^N (N \geqslant 2)$ acts isometrically. The commutative algebra $\mathcal{A}(M)$ is the algebra of functions $\mathcal{C}^\infty(M)$ with pointwise multiplication. With the scalar product induced by g, we can construct the Hilbert space $L^2(M, \mathrm{d}\mu_g)$ which can be decomposed as in (1). $\mathcal{A}(M) \equiv \mathcal{A}_0(M) \subset L^2(M, \mathrm{d}\mu_g)$ can then be twisted to $\mathcal{A}_\theta(M)$ using prescription (3). We now give examples of such twisted algebras.

2.1. The Moyal plane $\mathcal{A}_\theta(\mathbb{R}^{d+1})$

In this case, $\mathbb{R}^{d+1}$ acts on $\mathcal{A}(\mathbb{R}^{d+1}) = \mathcal{C}^\infty(\mathbb{R}^{d+1})$ by translations leaving the flat Euclidean metric invariant. The IRRs are labelled by the momentum $\lambda = (p^0, p^1, \ldots, p^d)$. A basis for $\mathcal{H}^{(p)}$ is formed by plane waves e_p with $e_p(x) = \mathrm{e}^{-\mathrm{i}p_\mu x^\mu}$, with $x = (x^0, x^1, \ldots, x^d)$ being a point of $\mathbb{R}^{d+1}$. Following (3), the *-product is defined by

$$e_p *_\theta e_q = e_p e_q\, \mathrm{e}^{-\frac{\mathrm{i}}{2} p_\mu \theta^{\mu\nu} q_\nu}. \tag{6}$$

This *-product defines the Moyal plane $\mathcal{A}_\theta(\mathbb{R}^{d+1})$.

2.2. Functions on tori

This example is a compact version of the Moyal plane. The manifold M is the torus $T^N (N \geqslant 2)$. The group $\mathbb{R}^N$ acts via its homomorphic image $U(1) \times U(1) \times \cdots \times U(1) \equiv U(1)^{\times N}$ on T^N leaving its flat metric invariant. The IRRs are labelled by the integral lattice $\mathbb{Z}^N$ with points $\lambda = (\lambda_1, \ldots, \lambda_N)$, $\lambda_i \in \mathbb{Z}$. A basis for $\mathcal{H}^{(\lambda)}$ is e_λ,

$$e_\lambda(p) = \mathrm{e}^{-\mathrm{i}\lambda_i \theta^i},$$

with $p = \left(\mathrm{e}^{\mathrm{i}\theta^1}, \ldots, \mathrm{e}^{\mathrm{i}\theta^N}\right)$ being a point of T^N. The *-product is

$$e_\lambda *_\theta e_{\lambda'} = e_\lambda e_{\lambda'}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\lambda_\mu \theta^{\mu\nu} \lambda'_\nu}. \tag{7}$$

It defines the noncommutative torus T^N_θ.

2.3. Functions on groups

Let $G = \{g\}$ be a simple, compact Lie group of rank $N \geqslant 2$ with an invariant measure $\mathrm{d}\mu$ and let T^N be its maximal torus. We can denote its IRRs by $\lambda = (\lambda_1, \ldots, \lambda_N)$, $\lambda_i \in \mathbb{Z}$ as before. T^N can act on G by left *or* right multiplication. Let us focus on the right action and the corresponding action on $L^2(G, \mathrm{d}\mu)$. As this action is unitary, we have decomposition (1). Now if $f_\lambda, f_{\lambda'} \in \mathcal{H}^{(\lambda),(\lambda')}$ are two smooth functions $f_\lambda, f_{\lambda'} \in \mathcal{C}^\infty(G)$, and $f_\lambda \otimes f_{\lambda'} \to f_\lambda f_{\lambda'}$ is their pointwise product, we can twist it as

$$f_\lambda *_\theta f_{\lambda'} = f_\lambda f_{\lambda'}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\lambda_\mu \theta^{\mu\nu} \lambda'_\nu}. \tag{8}$$

Now T^N acts on the right *and* left of G. That is, $T^N \times T^N$ acts on G and hence unitarily on $L^2(G, \mathrm{d}\mu)$. We can use any of its subgroups of rank $N \geqslant 2$ to perform the twist. With this generalization we can even choose $N = 1$ and twist $\mathcal{A}(S^3 \simeq SU(2))$, which is $\mathcal{C}^\infty(SU(2)) \simeq \mathcal{C}^\infty(S^3)$ with pointwise product.

We will be explicit about this twist. If $\{D^J_{\lambda_1,\lambda_2} | J \in \{0, \frac{1}{2}, 1, \ldots\}\}$ are the matrix elements of $SU(2)$ rotation matrices in the basis with the third component of the angular momentum diagonal, we have the expansion

$$f = \sum f^J_{\lambda_1,\lambda_2} D^J_{\lambda_1,\lambda_2}, \qquad f \in \mathcal{C}^\infty(G), \qquad f^J_{\lambda_1,\lambda_2} \in \mathbb{C} \tag{9}$$

by the Peter–Weyl theorem [12]. The twisted product is

$$D^J_{\lambda_1,\lambda_2} * D^K_{\lambda'_1,\lambda'_2} = D^J_{\lambda_1,\lambda_2} D^K_{\lambda'_1,\lambda'_2}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\lambda_\mu \theta^{\mu\nu} \lambda'_\nu}, \qquad \lambda = (\lambda_1, \lambda_2), \qquad \lambda' = (\lambda'_1, \lambda'_2). \tag{10}$$

2.4. Deforming graded algebras

Let a_λ and $a^\dagger_\lambda (\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N), N \geqslant 2, \lambda_i \in \mathbb{R})$ be the bosonic oscillators (for what follows, they can equally well be fermionic oscillators),

$$\left[a_\lambda, a^\dagger_{\lambda'}\right] = \delta_{\lambda\lambda'}, \qquad [a_\lambda, a_{\lambda'}] = \left[a^\dagger_\lambda, a^\dagger_{\lambda'}\right] = 0,$$

where δ is the Dirac delta and $\delta_{\lambda\lambda'} = \delta^{(N)}(\lambda - \lambda')$ if λ_i, λ'_j take continuous values. If we assign a charge λ to a_λ and $-\lambda$ to $a^\dagger_\lambda$, then $a_\lambda a_{\lambda'}$ and $a^\dagger_\lambda a^\dagger_{\lambda'}$ have charges $\lambda + \lambda'$ and $-\lambda - \lambda'$ while $a_\lambda a^\dagger_{\lambda'}$ and $a^\dagger_{\lambda'} a_\lambda$ have charges $\lambda - \lambda'$. Thus $a_\lambda, a^\dagger_{\lambda'}$ generate a graded algebra $\mathcal{A}$ with charge λ giving the grade

$$\mathcal{A} = \bigoplus \mathcal{A}^{(\lambda)}, \qquad \mathcal{A}^{(\lambda)}\mathcal{A}^{(\lambda')} \subseteq \mathcal{A}^{(\lambda+\lambda')}. \tag{11}$$

This feature allows us to twist $\mathcal{A}$ to an associative algebra $\mathcal{A}_\theta$ as before. Thus if $\alpha_\lambda \in \mathcal{A}^{(\lambda)}$, then

$$\alpha_\lambda *_\theta \alpha_{\lambda'} = \alpha_\lambda \alpha_{\lambda'}\, \mathrm{e}^{\frac{\mathrm{i}}{2}\lambda_\mu \theta^{\mu\nu} \lambda'_\nu}. \tag{12}$$

In particular,

$$\begin{aligned} a_\lambda *_\theta a_{\lambda'} &= a_\lambda a_{\lambda'}\, \mathrm{e}^{\frac{\mathrm{i}}{2}\lambda_\mu \theta^{\mu\nu} \lambda'_\nu}, \qquad & a^\dagger_\lambda *_\theta a^\dagger_{\lambda'} &= a^\dagger_\lambda a^\dagger_{\lambda'}\, \mathrm{e}^{\frac{\mathrm{i}}{2}\lambda_\mu \theta^{\mu\nu} \lambda'_\nu}, \\ a_\lambda *_\theta a^\dagger_{\lambda'} &= a_\lambda a^\dagger_{\lambda'}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\lambda_\mu \theta^{\mu\nu} \lambda'_\nu}, \qquad & a^\dagger_\lambda *_\theta a_{\lambda'} &= a^\dagger_\lambda a_{\lambda'}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\lambda_\mu \theta^{\mu\nu} \lambda'_\nu}. \end{aligned}$$

Note that $\mathcal{A}_\theta$ is graded:

$$\mathcal{A}_\theta = \bigoplus \mathcal{A}^{(\lambda)}_\theta.$$

Following the treatment of $\mathcal{A}(S^3)$, we can twist even oscillators with just one label $\lambda = \lambda_1$. In this case let λ_1 take values 1, 2 as an example so that a_i and $a^\dagger_i$ are the Schwinger oscillators for $SU(2)$. Arrange them as a matrix

$$\hat{g} = \begin{pmatrix} a_1 & -a^\dagger_2 \\ a_2 & a^\dagger_1 \end{pmatrix}, \tag{13}$$

$U(1) \times U(1)$ acts on $\hat{g}$

$$\hat{g} \to \begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}\varphi}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{\mathrm{i}\varphi}{2}} \end{pmatrix} \hat{g} \begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}\varphi}{2}} & 0 \\ 0 & \mathrm{e}^{-\frac{\mathrm{i}\varphi}{2}} \end{pmatrix}.$$

The $U(1) \times U(1)$ charges $q = (q_1, q_2)$ of $a_i, a^\dagger_j$ are

	a_1	a_2	$a^\dagger_1$	$a^\dagger_2$
$q =$	$(1, 1)$	$(-1, 1)$	$(-1, -1)$	$(1, -1)$

So we can label $a_i, a^\dagger_j$ by q:

$$a_1 = A_{(1,1)}, \qquad a_2 = A_{(-1,1)}, \qquad a^\dagger_1 = A^\dagger_{(-1,-1)}, \qquad a^\dagger_2 = A^\dagger_{(1,-1)},$$

the q-charge of $A^\dagger_q$ being $-q$.

The oscillators $A_q, A^\dagger_q$ are just like $a_\lambda, a^\dagger_\lambda$ for $N \geqslant 2$. Hence they can be twisted as previously.

3. $\mathcal{A}_\theta$ in terms of $\mathcal{A}_0$ and charges

In this section, we show how to realize $\mathcal{A}_\theta$ in terms of $\mathcal{A}$ and certain charge operators. This construction was used fruitfully in previous papers [7, 13].

The deformed product was previously given in terms of the undeformed product. It is also possible to give it as a relation between elements of $\mathcal{A}_\theta$ if $\mathcal{A}_0$ is commutative. Since

$$a_\lambda *_\theta a_{\lambda'} = a_\lambda a_{\lambda'}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\lambda_\mu\theta^{\mu\nu}\lambda'_\nu}, \qquad a_{\lambda'} *_\theta a_\lambda = a_{\lambda'} a_\lambda\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\lambda_\mu\theta^{\mu\nu}\lambda'_\nu},$$

then for Abelian $\mathbf{A}_0$, the relation is that of Weyl:

$$a_\lambda *_\theta a_{\lambda'} = \mathrm{e}^{\mathrm{i}\lambda_\mu\theta^{\mu\nu}\lambda'_\nu} a_{\lambda'} *_\theta a_\lambda.$$

Let us first discuss this simple case.

3.1. Deformations of Abelian algebras

Let Q_μ be the charge operator:

$$[Q_\mu, a_\lambda] = -\lambda_\mu a_\lambda. \tag{14}$$

Set

$$\hat{a}_\lambda = a_\lambda\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\lambda_\mu\theta^{\mu\nu}Q_\nu}. \tag{15}$$

Then under $\theta = 0$, unstarred products

$$\hat{a}_\lambda\hat{a}_{\lambda'} = a_\lambda a_{\lambda'}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}(\lambda+\lambda')_\mu\theta^{\mu\nu}Q_\nu}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\lambda_\mu\theta^{\mu\nu}\lambda'_\nu}. \tag{16}$$

Hence

$$\hat{a}_\lambda\hat{a}_{\lambda'} = \mathrm{e}^{\mathrm{i}\lambda_\mu\theta^{\mu\nu}\lambda_\nu}\hat{a}_{\lambda'}\hat{a}_\lambda \tag{17}$$

so that $\hat{a}_\lambda$s realize the $*_\theta$ algebra.

3.2. The case $\mathcal{A}_0$ is noncommutative

We now argue that $\mathcal{A}_\theta$ can always be realized as in section 3.1, also when $\mathcal{A}_0$ is noncommutative.

Let $\mathcal{A}_0 = \mathcal{A}$ be a possibly noncommutative graded algebra as above: $\mathcal{A}_0 = \oplus\mathcal{A}_0^{(\lambda)}$. With each $a_\lambda \in \mathcal{A}_0^{(\lambda)}$, we associate $\hat{a}_\lambda$ by the rule

$$\hat{a}_\lambda = a_\lambda\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\lambda_\mu\theta^{\mu\nu}Q_\nu}. \tag{18}$$

Then

$$\hat{a}_\lambda\hat{a}_{\lambda'} = \widehat{a_\lambda a_{\lambda'}}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\lambda_\mu\theta^{\mu\nu}\lambda'_\nu}. \tag{19}$$

Hence the image $\hat{\mathcal{A}}_0$ of $\mathcal{A}_0$ under the hat map is in fact closed under multiplication, that is, is an algebra. It is also graded:

$$\hat{\mathcal{A}}_0 = \bigoplus_\lambda \hat{\mathcal{A}}_0^{(\lambda)} \tag{20}$$

$$\hat{\mathcal{A}}_0^{(\lambda)} = \text{Image under hat of } \mathcal{A}_0. \tag{21}$$

It is in fact a subalgebra of $\mathcal{A}_\theta$ since we also have (17) from (19).

Conversely, if we define a_λ by $a_\lambda = \hat{a}_\lambda\,\mathrm{e}^{\frac{\mathrm{i}}{2}\lambda_\mu\theta^{\mu\nu}Q_\nu}$, then a_λ's fulfil the relations of $\mathcal{A}_0$. The hat map being invertible, we conclude that $\hat{\mathcal{A}}_0 = \mathcal{A}_\theta$ and that the inverse map $\hat{a}_\lambda \to a_\lambda$ gives $\mathcal{A}_0$.

4. Kac–Moody algebras

Now we discuss $SU(N)$ KM algebras. At the first instance, we assume that $N \geqslant 3$, so that the rank of $SU(N)$ is at least 2. The (complexified) Lie algebra of $SU(N)$ has a basis H_i, $E_{\pm\alpha}$, where H_i spans the Cartan subalgebra and $E_{\pm\alpha}$ are the raising and lowering operators:

$$[H_i, H_j] = 0, \qquad [H_i, E_\alpha] = \alpha_i E_\alpha, \qquad i, j \in \{1, 2, \ldots, N-1\}.$$

There is an oscillator construction of the KM algebra. We can twist the oscillators which deform the KM algebra. Or we can directly deform the KM algebra. These sets of deformations give different deformed KM algebras as we shall see.

4.1. Oscillator twists

Let $a_i, a_j^\dagger (1 \leqslant i, j \leqslant N)$ be the bosonic annihilation and creation operators. Then if $\lambda_a (a = 1, 2, \ldots, N^2 - 1)$ are the $N \times N$ Gell–Mann matrices of $SU(N)$, we have the Schwinger construction of the $SU(N)$ Lie algebra generators on the Fock space:

$$\Lambda_\alpha = a^\dagger \lambda_\alpha a \qquad [\Lambda_a, \Lambda_b] = \mathrm{i} f_{abc} \Lambda_c. \tag{22}$$

We can express H_i, E_α in terms of Λ_a in a well-known way.

We can also label $a_j^\dagger$'s by weights $\mu^{(j)}$ such that

$$\left[H_i, a^\dagger_{\mu^{(j)}}\right] = \mu_i^{(j)} a^\dagger_{\mu^{(j)}}. \tag{23}$$

Then a_j has weight $-\mu^{(j)}$. We can write it as $a_{\mu^{(j)}}$, using the negative of the weights as the subscript for a's.

For the KM realization, we need infinitely many such oscillator pairs $a^{(n)\dagger}_{\mu^{(j)}}, a^{(n)}_{\mu^{(j)}} (n = 0, 1, \ldots)$, $a^{0\dagger}_{\mu^{(j)}}, a^{0}_{\mu^{(j)}}$ being $a^\dagger_{\mu^{(j)}}, a_{\mu^{(j)}}$. Their weights are $\pm\mu^{(j)}$

$$\left[H_i, a^{(n)\dagger}_{\mu^{(j)}}\right] = \mu_i^{(j)} a^{(n)\dagger}_{\mu^{(j)}}, \qquad \left[H_i, a^{(n)}_{\mu^{(j)}}\right] = -\mu_i^{(j)} a^{(n)}_{\mu^{(j)}}.$$

As $SU(N)$ now acts on all the oscillators, the new Λ_a are

$$\Lambda_a = \sum_{n \geqslant 0} a^{(n)\dagger} \lambda_a a^{(n)}. \tag{24}$$

We can next do the twist

$$a^{(n)\dagger}_{\mu^{(j)}}, a^{(n)}_{\mu^{(j)}} \longrightarrow \hat{a}^{(n)\dagger}_{\mu^{(j)}}, \hat{a}^{(n)}_{\mu^{(j)}},$$

where

$$\hat{a}^{(n)\dagger}_{\mu^{(j)}} = a^{(n)\dagger}_{\mu^{(j)}} \,\mathrm{e}^{\frac{\mathrm{i}}{2}\mu_k^{(j)} \theta^{kl} H_l}, \tag{25}$$

$$\hat{a}^{(n)}_{\mu^{(j)}} = a^{(n)}_{\mu^{(j)}} \,\mathrm{e}^{-\frac{\mathrm{i}}{2}\mu_k^{(j)} \theta^{kl} H_l}. \tag{26}$$

Observe that H_l's form a set of charge operators, i.e., $[H_i, H_j] = 0$, for all $i, j = 1, \ldots, N-1$.

4.2. The Kac–Moody deformations

As mentioned above, there are two ways to deform the KM algebra. The first deformation is induced by those of the oscillators. In the second, we deform the KM generators directly. They lead to different deformations of the KM algebra.

4.2.1. From twisted oscillators. The untwisted bosonic (or fermionic) oscillators give the KM generators

$$J_a^{(n)} = \sum_m a^{(n+m)\dagger}\lambda_a a^{(m)} \equiv \sum_m a^{(n+m)\dagger}_{\mu^{(j)}}(\lambda_a)_{jk} a^{(m)}_{\mu^{(k)}}, \tag{27}$$

$$J_a^0 \equiv \Lambda_a, \qquad a^{(r)}, a^{(r)^\dagger} = 0 \quad \text{if} \quad r < 0. \tag{28}$$

We write the deformed KM generators $\hat{J}_a^{(n)}$ in terms of the twisted bosonic oscillators (25) and (26), so that

$$\hat{J}_a^{(n)} = \sum_m \hat{a}^{(n+m)\dagger}_{\mu^{(j)}}(\lambda_a)_{jk}\hat{a}^{(m)}_{\mu^{(k)}}. \tag{29}$$

(The discussion here remains valid also with the twisted fermionic oscillators.) Substituting (25) and (26), we find

$$\begin{aligned}\hat{J}_a^{(n)} &= \sum_m a^{(n+m)\dagger}_{\mu^{(j)}}(\lambda_a)_{jk} a^{(m)}_{\mu^{(k)}}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\mu_p^{(j)}\theta^{pq}\mu_q^{(k)}}\, \mathrm{e}^{\frac{\mathrm{i}}{2}(\mu^{(j)}-\mu^{(k)})_p\theta^{pq}H_q} \\ &= \sum_m a^{(n+m)\dagger}_{\mu^{(j)}}\left\{(\lambda_a)_{jk}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{ad}\overleftarrow{H}_p\theta^{pq}\overrightarrow{H}_q}\right\} a^{(m)}_{\mu^{(k)}}\, \mathrm{e}^{\frac{\mathrm{i}}{2}(\mu^{(j)}-\mu^{(k)})_p\theta^{pq}H_q},\end{aligned} \tag{30}$$

where $\mathrm{ad}H_p\cdot = [H_p, \cdot]$ is the adjoint action of H_p.

The exponential in the braces acts only on oscillators in the manner already shown. The Gell–Mann matrices are thus effectively changed to the operators

$$\hat{\lambda}_a = \lambda_a\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{ad}\overleftarrow{H}_p\theta^{pq}\overrightarrow{H}_q}. \tag{31}$$

We note that

$$\left[\lambda_a, \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{ad}\overleftarrow{H}_p\theta^{pq}\overrightarrow{H}_q}\right] = 0. \tag{32}$$

Hence

$$[\hat{\lambda}_a, \hat{\lambda}_b] = \mathrm{i}C_{ab}{}^c\hat{\lambda}_c\, \mathrm{e}^{-\mathrm{i}\mathrm{ad}\overleftarrow{H}_p\theta^{pq}\mathrm{ad}\overrightarrow{H}_q}. \tag{33}$$

Thus, we can write

$$\hat{J}_a^{(n)} = \sum_m a^{(n+m)\dagger}_{\mu^{(j)}}(\hat{\lambda}_a)_{jk} a^{(m)}_{\mu^{(k)}}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{ad}\overleftarrow{H}_p\theta^{pq}H_q}. \tag{34}$$

Now

$$\begin{aligned}\hat{J}_a^{(n)}\hat{J}_b^{(n')} = \sum_{\substack{m,m'\\ j,j'\\ k,k'}} &\left(a^{(n+m)\dagger}_{\mu^{(j)}}(\hat{\lambda}_a)_{jk}a^{(m)}_{\mu^{(k)}}\right)\left(a^{(n'+m')\dagger}_{\mu^{(j')}}(\hat{\lambda}_b)_{j'k'}a^{(m')}_{\mu^{(k')}}\right) \\ &\times \mathrm{e}^{\frac{\mathrm{i}}{2}(\mu^{(j)}-\mu^{(k)})_p\theta^{pq}(\mu^{(j')}-\mu^{(k')})_q} \times \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{ad}\overleftarrow{H}_p\theta^{pq}H_q},\end{aligned} \tag{35}$$

where $\mathrm{ad}\overleftarrow{H}_p$ is to be applied to all four oscillators, so that

$$\begin{aligned}\hat{J}_a^{(n)}\left(\mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{ad}\overleftarrow{H}_p\theta^{pq}\mathrm{ad}\overrightarrow{H}_q}\right)\hat{J}_b^{(n')} = \sum_{\substack{m,m'\\ j,j'\\ k,k'}} &\left(a^{(n+m)\dagger}_{\mu^{(j)}}(\hat{\lambda}_a)_{jk}a^{(m)}_{\mu^{(k)}}\right)\left(a^{(n'+m')\dagger}_{\mu^{(j')}}(\hat{\lambda}_b)_{j'k'}a^{(m')}_{\mu^{(k')}}\right) \\ &\times \mathrm{e}^{-\frac{\mathrm{i}}{2}\mathrm{ad}\overleftarrow{H}_p\theta^{pq}H_q}.\end{aligned} \tag{36}$$

From this follows the deformed KM algebra

$$\hat{J}_a^{(n)}\left(\mathrm{e}^{\mathrm{iad}\overleftarrow{H}_p\theta^{pq}\mathrm{ad}\overrightarrow{H}_q}\right)\hat{J}_b^{(n')} - \hat{J}_b^{(n')}\left(\mathrm{e}^{\mathrm{iad}\overleftarrow{H}_p\theta^{pq}\mathrm{ad}\overrightarrow{H}_q}\right)\hat{J}_a^{(n)}$$
$$= \mathrm{i}C_{ab}{}^c\hat{J}_c^{(n+n')} + kn\delta_{ab}\delta^{n+n',0}\,\mathrm{e}^{-\mathrm{i}(\mu^{(j)}+\mu^{(j')})_p\theta^{pq}(\mu^{(k)}+\mu^{(k')})_q}, \tag{37}$$

where k, being the level of the KM algebra, is 1 for the oscillator construction. Observe that in this case the commutator defining the KM algebra has been deformed.

4.2.2. From direct deformation of KM generators. We can express the KM algebra in a new basis where λ_a are exchanged for H_i, E_α (in the defining representation). This then gives KM generators $SU(N)$ roots. Roots being special instances of weights, we can also write the basis as $J_i^{(n)}$, $J_{\mu^{(s)}}^{(n)}$. Now we dispense with oscillators and consider any level k of the KM algebra. According to our prescriptions, the deformed siblings of the KM basis elements are

$$\begin{aligned}\tilde{J}_i^{(n)} &= J_i^{(n)} \\ \tilde{J}_{\mu^{(s)}}^{(n)} &= J_{\mu^{(s)}}^{(n)}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\mu_p^{(s)}\theta^{pq}H_q}.\end{aligned} \tag{38}$$

If we put $E_{\mu^{(s)}}$ for λ_a, with $\mu^{(s)}$ being a root, then it has nonvanishing matrix elements only for $\mu^{(j)} - \mu^{(k)} = \mu^{(s)}$. So (38) is almost the same as (30), but not quite. Equation (30) has the extra phase

$$\mathrm{e}^{\frac{\mathrm{i}}{2}\mu_p^{(j)}\theta^{pq}\mu_q^{(k)}}$$

inside the sum. Substituting $\mu^{(j)} = \mu^{(k)} + \mu^{(s)}$ and using the antisymmetry of θ^{pq}, this simplifies to

$$\mathrm{e}^{\frac{\mathrm{i}}{2}\mu_p^{(s)}\theta^{pq}\mu_q^{(k)}}.$$

So it appears that the two deformations are different.

The algebraic structure of $\tilde{J}_{\mu^{(s)}}^{(n)}$ is simple. Clearly

$$\left[\tilde{J}_i^{(n)}, \tilde{J}_j^{(m)}\right] = kn\delta_{ij}\delta^{n+m,0} \tag{39}$$

$$\left[\tilde{J}_i^{(n)}, \tilde{J}_{\mu^{(s)}}^{(m)}\right] = \mu_i^{(s)}\tilde{J}_{\mu^{(s)}}^{(n+m)}. \tag{40}$$

Also

$$\left[\tilde{J}_{\mu^{(s)}}^{(n)}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\mu_p^{(s)}\theta^{pq}H_q}, \tilde{J}_{\mu^{(t)}}^{(m)}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\mu_{p'}^{(t)}\theta^{p'q'}H_{q'}}\right] = \left[J_{\mu^{(s)}}^{(n)}, J_{\mu^{(t)}}^{(m)}\right] = N_{st}J_{\mu^{(s)}+\mu^{(t)}}^{(n+m)}, \tag{41}$$

where

$$N_{st} \neq 0 \quad\Longleftrightarrow\quad \mu^{(s)} + \mu^{(t)} \text{ is a root.}$$

Hence, the LHS of (41) is equal to

$$N_{st}\tilde{J}_{\mu^{(s)}+\mu^{(t)}}^{(n+m)}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}\left(\mu^{(s)}+\mu^{(t)}\right)_p\theta^{pq}H_q}. \tag{42}$$

4.2.3. Hopf structure of deformations. The existence of a coproduct Δ is an essential property of a symmetry algebra. With the help of Δ, we can compose subsystems transforming by the symmetry algebra and define the action of the latter on the composite system. An algebra with a coproduct Δ is known as a coalgebra.

If the symmetry algebra has a more refined structure and is Hopf (and not just a coalgebra), then it has all the essential features of a group. In this case, we can regard it as a 'quantum group of symmetries' generalizing 'classical' symmetry groups [14].

General deformations of a Hopf algebra such as the KM algebra need not preserve its Hopf structure. We now show that $\tilde{J}^{(m)}_{\mu^{(s)}}$ is in fact a basis of generators for a Hopf algebra. The situation as regards $\hat{J}^{(n)}_{a}$ is less clear.

Both $J^{(m)}_{\mu^{(s)}}$ and H_q are elements of a Hopf algebra. In fact $J^{(0)}_{\mu^{(s)}}$ and H_q generate (complexified) $su(N)$ Lie algebra. From the expression for $\tilde{J}^{(m)}_{\mu^{(s)}}$, we see that $\tilde{J}^{(m)}_{\mu^{(s)}}$ is also an element of the same Hopf algebra establishing the claim. If Δ is the coproduct, we can write

$$\Delta\big(\tilde{J}^{(m)}_{\mu^{(s)}}\big) = \Delta\big(J^{(m)}_{\mu^{(s)}}\big)\, \mathrm{e}^{\frac{\mathrm{i}}{2}\mu^{(s)}_p \theta^{pq} \Delta(H_q)}, \tag{43}$$

Δ on KM generators having familiar expressions such as

$$\Delta(H_q) = \mathbf{1} \otimes H_q + H_q \otimes \mathbf{1}.$$

As regards $\hat{J}^{(n)}_{a}$, we do not have an answer. They do not seem to be elements of the enveloping algebra of the KM algebra.

4.2.4. Remarks. In both the oscillator and KM deformations, there is a superscript such as (n) identifying the mode. It is passive in the process of deformation: the antisymmetric deformation matrix $\theta = (\theta^{\mu\nu})$ is independent of n.

We can make it depend on n. We can replace θ by $\theta^{(n)} = \theta^{(n)}_{\mu\nu}$ in the preceding construction, thereby obtaining very general deformations. We will not study this generalization in this paper.

We can also deform $SU(2)$ KM and its Virasoro algebras by twisting the Schwinger oscillators of $SU(2)$ following section 2.4. This leads to the deformed $SU(2)$ KM currents similar to $\hat{J}^{(m)}_{\mu^{(s)}}$ above.

4.3. Virasoro algebra

The Virasoro algebra can be realized from the KM algebra by the Sugawara construction. Its generators L_n can be written as

$$L_n = \frac{1}{2k + C_N} : J^{(m+n)}_a J^{(-m)}_a :, \tag{44}$$

where k is the level of the KM algebra and C_N is the eigenvalue of the Casimir operator of $SU(N)$.

Here : : denotes normal ordering with regard to the currents J_a. Those with the positive superscripts stand to the right and we have reverted to the original subscripts a. The deformed currents on the RHS of (44) will then deform the Virasoro algebra as well.

The central role of the Virasoro algebra in physics is as a symmetry algebra. This suggests that its deformation from $\tilde{J}^{(n)}_{a}$ is more interesting. So we focus on the deformation from $\tilde{J}^{(n)}_{\mu^{(s)}}$. They deform L_n to

$$\tilde{L}_n = \frac{1}{2k + C_N} \sum_m : \tilde{J}^{(m+n)}_a \tilde{J}^{(-m)}_a : .$$

Using (38), we can write

$$\tilde{L}_n = \frac{1}{2k + C_N} \sum_m : J^{(m+n)}_{\mu^{(s)}} J^{(-m)}_{-\mu^{(s)}} + J^{(m+n)}_i J^{(-m)}_i :, \tag{45}$$

where we used $\mu^{(s)}_p \theta^{pq}\big(-\mu^{(s)}_q\big) = 0$. Thus, in this approach the Virasoro algebra is not deformed at all.

For an implementation of the quantum conformal invariance in the $2 - d$ Moyal plane see [15].

5. On statistics

Suppose we have a free quantum scalar field φ on the commutative manifold $\mathbb{R}^{d+1}$ with the Fourier expansion

$$\varphi(x) = \int \mathrm{d}\mu(p)[c(p)e_p(x) + c^\dagger(p)e_{-p}(x)],$$

$$p\cdot x = p_0x_0 - \vec{p}\cdot\vec{x}, \qquad e_p(x) = \mathrm{e}^{-\mathrm{i}p\cdot x}, \qquad \mathrm{d}\mu(p) = \frac{1}{(2\pi)^{\frac{d}{2}}}\frac{d^d p}{2|p_0|}, \qquad p_0 = \sqrt{\vec{p}^2 + m^2},$$

where x_0 and $\vec{x}$ are the time and space coordinates, and m is the mass of φ. The creation and annihilation operators fulfil the standard commutation relations

$$[c(p), c^\dagger(p')] = 2|p_0|\delta^d(p'-p), \qquad [c^\dagger(p), c^\dagger(p')] = [c(p), c(p')] = 0.$$

We can now twist $c(p)$ and $c^\dagger(p)$ to

$$a(p) = c(p)\,\mathrm{e}^{-\frac{\mathrm{i}}{2}p_\mu\theta^{\mu\nu}P_\nu}, \qquad a^\dagger(p) = c^\dagger(p)\,\mathrm{e}^{\frac{\mathrm{i}}{2}p_\mu\theta^{\mu\nu}P_\nu}, \tag{46}$$

where p_0 and $\vec{p}$ are the energy and the momentum, respectively, and P_μ is the translation operator,

$$P_\mu := \int \mathrm{d}\mu(p)p_\mu c^\dagger(p)c(p) = \int \mathrm{d}\mu(p)p_\mu a^\dagger(p)a(p), \tag{47}$$

$$[P_\mu, a^\dagger(p)] = p_\mu a^\dagger(p), \qquad [P_\mu, a(p)] = -p_\mu a(p). \tag{48}$$

The P_μ operator is the analogue of Q_μ. It was studied in [7, 8, 13, 16]. The twist of c's twists statistics since a's and $a^\dagger$'s no longer fulfil standard relations:

$$a(p)a(p') = a(p')a(p)\,\mathrm{e}^{\mathrm{i}p_\mu\theta^{\mu\nu}p'_\nu}, \qquad a^\dagger(p)a^\dagger(p') = a^\dagger(p')a^\dagger(p)\,\mathrm{e}^{\mathrm{i}p_\mu\theta^{\mu\nu}p'_\nu},$$
$$a(p)a^\dagger(p') = 2|p_0|\delta^d(p-p') + a^\dagger(p')a(p)\,\mathrm{e}^{-\mathrm{i}p_\mu\theta^{\mu\nu}p'_\nu}.$$

The implication of this twist is that the n-particle wavefunction $\psi_{k_1\cdots k_n}$,

$$\psi_{k_1\cdots k_n}(x_1,\ldots,x_n) = \langle 0|\,\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)a^\dagger_{k_n}a^\dagger_{k_{n-1}}\cdots a^\dagger_{k_1}\,|0\rangle\,,$$

is not symmetric under the interchange of k_i. Rather it fulfils a twisted symmetry:

$$\psi_{k_1\cdots k_i k_{i+1}\cdots k_n} = \mathrm{e}^{-\mathrm{i}k_{i,\mu}\theta^{\mu\nu}k_{i+1,\nu}}\psi_{k_1\cdots k_{i+1}k_i\cdots k_n}. \tag{49}$$

This twisted statistics by the following chain of connections implies that spacetime is the Moyal plane with

$$e_p *_\theta e_{p'} = \mathrm{e}^{-\frac{\mathrm{i}}{2}p_\mu\theta^{\mu\nu}p'_\nu}e_{p+p'}. \tag{50}$$

The chain is as follows: let g be an element of the Lorentz group without time reversal. For $\theta^{\mu\nu} = 0$, it acts on $\psi_{k_1\cdots k_n}$ by the representative $g\otimes g\otimes\cdots\otimes g$ (n factors) compatibly with the symmetry of $\psi_{k_1\cdots k_n}$. This action is based on the coproduct

$$\Delta_0(g) = g\otimes g.$$

But for $\theta^{\mu\nu}\neq 0$, and for $g\neq$ identity, already for $n=2$,

$$\Delta_0(g)\psi_{k_1,k_2} = \psi_{gk_1,gk_2} = \mathrm{e}^{-\mathrm{i}k_{1,\mu}\theta^{\mu\nu}k_{2,\nu}}\Delta_0(g)\psi_{k_2,k_1}$$
$$= \mathrm{e}^{-\mathrm{i}k_{1,\mu}\theta^{\mu\nu}k_{2,\nu}}\psi_{gk_2,gk_1} \neq \mathrm{e}^{-\mathrm{i}(gk_1)_\mu\theta^{\mu\nu}(gk_2)_\nu}\psi_{gk_2,gk_1}.$$

Thus, the naive coproduct Δ_0 is incompatible with the statistics (49). It has to be twisted to

$$\Delta_\theta(g) = F_\theta^{-1}(g\otimes g)F_\theta, \qquad F_\theta = \mathrm{e}^{\frac{\mathrm{i}}{2}\partial_\mu\otimes\theta^{\mu\nu}\partial_\nu} \tag{51}$$

for such compatibility.

But then Δ_θ is incompatible with the commutative multiplication map m_0:

$$m_0(e_p \otimes e_{p'}) = e_{p+p'}.$$

That is,

$$m_0[\Delta_\theta(g)(e_p \otimes e_{p'})] \neq g e_{p+p'}.$$

We are forced to change m_0 to

$$m_\theta = m_0 F_\theta$$

for this compatibility, that is, to preserve spacetime symmetries as automorphisms. Since

$$m_\theta(e_p \otimes e_{p'}) = e_p * e_{p'} \equiv \mathrm{e}^{-\frac{\mathrm{i}}{2} p_\mu \theta^{\mu\nu} p'_\nu} e_{p+p'}, \tag{52}$$

we end up with the Moyal plane.

Thus, statistics can lead to spacetime noncommutativity. This idea is being studied further by our group.

In general, when we twist the creation and annihilation operators, such as $a_\lambda^\dagger$ and a_λ, then we twist statistics just as in the Moyal case (50). The spatial slice associated with these operators can be the N-torus T^N if $\lambda_i \in \mathbb{Z}$, with coordinates $(\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}, \ldots, \mathrm{e}^{\mathrm{i}\theta_N})$. The field operator at a fixed time is then φ where

$$\varphi(\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}, \ldots, \mathrm{e}^{\mathrm{i}\theta_N}) = \sum_{\{\lambda_1\}} \left[a_\lambda \, \mathrm{e}^{-\mathrm{i}\sum_i \lambda_i \theta_i} + a_\lambda^\dagger \, \mathrm{e}^{\mathrm{i}\sum_i \lambda_i \theta_i}\right], \tag{53}$$

where we have assumed for simplicity that $\varphi^\dagger = \varphi$. Then the torus algebra is twisted for the same reason as in the Moyal case. If e_λ denotes the function with values

$$e_\lambda(\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}, \ldots, \mathrm{e}^{\mathrm{i}\theta_N}) = \mathrm{e}^{-\mathrm{i}\sum_i \lambda_i \theta_i},$$

then their product $*$ is defined by

$$e_\lambda * e_{\lambda'} = \mathrm{e}^{\mathrm{i}\lambda_\mu \theta^{\mu\nu} \lambda'_\nu} e_{\lambda+\lambda'}. \tag{54}$$

That is, we get back the twisted $\mathcal{C}^\infty(T^N)$ algebra of (7).

If there is a collection of oscillators indexed by n as in section 4.1, or equally KM generators with an index n, it is more reasonable to regard them as associated $\mathcal{C}^\infty(S^1)$. For example,

$$J_a(\theta) = \sum J_a^{(n)} e_n(\mathrm{e}^{\mathrm{i}\theta}), \qquad e_n(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{-\mathrm{i}n\theta}.$$

This expansion is the known one for the generators of the Lie algebra of the centrally extended loop group.

In this case, a becomes an internal index. There is perhaps still an interpretation of the deformation in terms of the statistics of 'internal' excitations associated with a. But $\mathcal{C}^\infty(S^1) = \mathcal{C}^\infty(T^1)$ cannot be deformed like $\mathcal{C}^\infty(T^N)$ for $N \geqslant 2$. So what these deformations have to do with spacetime twists is not clear.

6. Final remarks

As remarked earlier, deformations such as those we consider appeared first in the quantum group theory. Recently, they found concrete applications in discussions of quantum theories on the Moyal plane and in particular Pauli principle violations and the absence of UV–IR mixing [7, 8, 13]. Further applications exist. The twists of the Moyal plane are those of the worldvolume. We can also twist the target of fields with striking results. Work on such twists is now being written up [17].

As mentioned earlier, recently, Fairlie and Zachos proposed an 'atavistic' algebra [9], which is based on the oscillator algebra. They also called attention to the possible quantum field theoretical applications of their algebra.

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